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ONE-DIMENSIONAL TWO-FLUID EQUATIONS FOR HORIZONTAL STRATIFIED TWO-PHASE FLOW

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Abstract—Advanced computer codes for water reactor loss-of-coolant analysis are based on the use of the two-fluid model of two-phase flow, in which conservation equations are solved for the gas and liquid phases separately. The standard two-fluid equations, however, sometimes predict the growth of instabilities in the flow, and occasionally become improperly posed. These difficulties have in the past led to the proposal of several different forms for the conservations equations.

To help resolve these uncertainties a widely accepted form of the one-dimensional two-fluid equations is used to calculate wave propagation speeds, and stability limits, for the illustrative case of a frictionless horizontal stratified gas–liquid flow. Calculated propagation velocities are shown to agree with the appropriate limit of an exact solution, and the predicted stability limits are found consistent with available observations on the stability of the stratified flow regime.

These comparisons help improve confidence in the ability of the two-fluid equations to analyse more complex problems in transient two-phase flow.

1. INTRODUCTION

Models used to analyse postulated loss-of-coolant accidents in water cooled nuclear reactors generally treat the flowing two-phase mixture as homogeneous, or invoke external correlations to account for effects of relative motion between the phases. In advanced computer codes, however, increasing use is being made of the two-fluid or separated-flow formulation, in which the conservation equations are solved for the phases separately (Liles *et al.* 1978). Use of the two-fluid model enables the non-equilibrium exchange of heat mass and momentum between the phases to be more easily included in the analysis.

A recurring problem in using the two-fluid method arises because commonly used forms of the conservation equations frequently predict instabilities which are not experimentally observed. In some cases the instabilities occur even for high frequency perturbations and the resulting unbounded growth rates imply that the conservation equations are ill-posed,† in the sense that a unique solution does not exist for a given set of initial and boundary conditions (Gidaspow 1974, Ramshaw & Trapp 1978). For strongly coupled flows (e.g. bubble flow) the problem of instability and illposedness can be removed by allowing for the presence of inertial interactions between the phases (Stuhmiller 1977). However for other flow regimes (e.g. stratified flows) it is not obvious that such interaction terms are present. This problem has caused a great deal of confusion in the literature, leading some authors to suggest that momentum exchange terms must have been omitted from the conservation equations. Various "corrected" equation sets have been proposed along these lines (see e.g. Lyczkowski *et al.* 1975, Agee *et al.* 1978, Mathers *et al.* 1978).

In this paper we examine the ability of conventional one-dimensional two-fluid equations to describe an illustrative example of a horizontal stratified flow in which viscosity is ignored, but effects of gravity and surface tension are retained. The velocity of small amplitudes waves predicted by the equations is compared with the exact solution that can be obtained for this case. Implied stability limits are then discussed, and compared with experimental observations on the breakdown of stable stratified flows in horizontal ducts.

†A discussion of the conditions for, and implications of, ill-posed equations is given by Richtmyer & Morton (1976).

2. ONE-DIMENSIONAL TWO-FLUID MODEL FOR STRATIFIED FLOW

We first give one-dimensional two-fluid equations appropriate for a horizontal stratified flow. For simplicity we consider a duct of rectangular cross-section (see figure 1) and ignore dissipative effects of viscosity and interphase mass and heat transfer. The stability and well-posedness of the conservation equations is examined by considering the motion of small amplitude waves. Predicted wave velocities are compared with results of exact two-dimensional calculations in section 3, and predicted stability limits compared with data in section 4.

2.1 *Derivation of conservation equations*

The two-fluid equations for conservation of mass momentum and evergy are derived by Ishii (1975) for a general three-dimensional two-phase flow. One-dimensional forms can be obtained by integrating across the duct area. Neglecting viscous interactions, and for the moment ignoring pressure changes at the interface due to surface tension, the equations for mass and momentum conservation for phase j so obtained are (for horizontal flow)

$$\left.\begin{array}{c}\frac{\partial}{\partial t}(\alpha_j\bar{\rho}_j)+\frac{\partial}{\partial x}(\alpha_j\rho_j\bar{U}_j)=0\\ \alpha_j\bar{\rho}_j\frac{\partial \bar{U}_j}{\partial t}+\alpha_j\bar{\rho}\bar{U}_j\frac{\partial \bar{U}_j}{\partial x}+\alpha_j\frac{\partial \bar{P}_j}{\partial x}-(\bar{P}^*-\bar{P}_j)\frac{\partial \alpha_j}{\partial x}=0\end{array}\right\}. \qquad [1]$$

Here x is the spatial coordinate along the duct axis and t is time. ρ_j, P_j and U_j denote respectively the density, pressure and velocity of phase j and P^* is the interface pressure. The over-bars refer to phase average quantities defined by:

$$\bar{a}_j = A_j^{-1}\int_{A_j} a_j \,\mathrm{d}A .$$

$\alpha_j = A_j/A$ is the fraction of the duct area occupied by phase j. Note that (i) averages of products have been taken as equal to products of averages and (ii) all flow variables represent mean values over some short time interval. In the following j refers to either the gas phase $j \equiv G$ or the liquid phase $j \equiv L$.

Equivalent equations to [1] have been derived in spatially averaged form by Agee *et al.* (1978).

Gravity does not occur explicitly in [1], but influences the average phase pressures through the hydrostatic effect. In the gas phase we have, for the present stratified system:

$$P_G = P^* - y\rho_G g,$$

where y is the distance above the interface and g is the acceleration due to gravity. It follows that, for the present rectangular duct:

$$\bar{P}_G = P^* - \tfrac{1}{2}\alpha_G\rho_G g H, \qquad [2]$$

where H is the duct height.

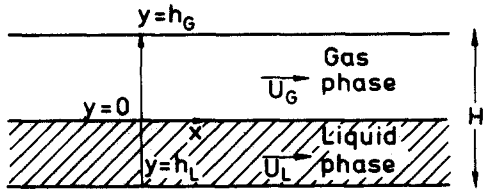

Figure 1. Horizontal stratified flow in a rectangular duct.

The corresponding result for the liquid phase is

$$\bar{P}_L = P^* + \tfrac{1}{2}\alpha_L\rho_L gH. \qquad [2']$$

Substituting for $\bar{P}_j$ in [1] using [2] and [2'] we get

$$\left.\begin{aligned} &\frac{\partial}{\partial t}(\alpha_j\bar{\rho}_j) + \frac{\partial}{\partial x}(\alpha_j\bar{\rho}_j\bar{U}_j) = 0 \\ &\alpha_j\bar{\rho}_j\frac{\partial\bar{U}_j}{\partial t} + \alpha_j\bar{\rho}_j\bar{U}_j\frac{\partial\bar{U}_j}{\partial x} + \alpha_j\frac{\partial P^*}{\partial x} \pm \alpha_j\bar{\rho}_j gH\frac{\partial\alpha_j}{\partial x} = 0 \end{aligned}\right\}, \qquad [3]$$

where the positive and negative signs are appropriate for the liquid and gas phases respectively.

2.2 *Propagation analysis*

We consider the motion of small amplitude waves of angular frequency ω and wavelength k travelling along the duct axis. In the disturbed flow oscillations in the primary variables are given by

$$\psi = \psi_0 + \psi' \exp i(\omega t - kx), \qquad [4]$$

where $\psi \equiv P^*, \rho_j, \alpha_j, \bar{U}_j$. ψ_0 denotes the value of ψ in the undisturbed flow and $\psi' \ll \psi_0$. Variations in density are related to pressure variations through

$$\rho_j' = P_j'/c_j^2, \qquad [5]$$

where c_j is the sound speed in phase j.

In the undisturbed flow the interface is assumed to be horizontal (see figure 1) so that $\alpha_{G0} = h_G/H$, $\alpha_{L0} = h_L/H$.

Substituting the perturbed forms [4] into the conservation equations [3] and using [5], neglecting terms of higher than first order in the primed quantities, we get a set of homogeneous linear simultaneous equations in the primed quantities. The secular equation for this set, which is the required dispersion relation, has the form

$$h_G\rho_*\mu_G^2\omega_L^2 + h_L\mu_L^2\omega_G^2 - gk^2h_Lh_G(\rho_*\mu_G^2 - \mu_L^2) = 0, \qquad [6]$$

where $\mu_j^2 = (k^2 - \omega_j^2/c_j^2)$ and $\omega_j = (\omega - U_{j0}k)$. ρ_* is the density ratio, ρ_L/ρ_G.

Equation [6] forms a quartic equation for the phase velocity ω/k (which is frequency independent). The four roots correspond to two surface waves (gravity waves) and two pressure waves. A necessary condition for these roots to be real is that the discriminant of the quartic, Δ, is positive (see Turnbull 1952). For [6] this condition may be written as

$$\Delta = I^3 - 27J^2 > 0, \qquad [7]$$

where

$$I = \frac{1}{12}(U_{GL}^2 - \beta_1 - \beta_2)^2 + \gamma,$$

$$J = \tfrac{1}{4}U_{GL}^2\beta_1\beta_2 - \frac{1}{6^3}(U_{GL}^2 - \beta_1 - \beta_2)^3 + \gamma\left\{\frac{1}{6}(U_{GL}^2 - \beta_1 - \beta_2) - \frac{U_{GL}^2}{4}\right\},$$

and

$$\beta_1 = h_G\rho_L\left(1 - \frac{gh_L\rho_G}{\rho_L c_L^2}\right)\left(\frac{h_G\rho_L}{c_G^2} + \frac{h_L\rho_G}{c_L^2}\right)^{-1},$$

$$\beta_2 = h_L\rho_G\left(1 + \frac{gh_G\rho_L}{\rho_G c_G^2}\right)\left(\frac{h_G\rho_L}{c_G^2} + \frac{h_L\rho_G}{c_L^2}\right)^{-1},$$

$$\gamma = gh_Gh_L(\rho_L - \rho_G)\left(\frac{h_G\rho_L}{c_G^2} + \frac{h_L\rho_G}{c_L^2}\right)^{-1}.$$

When $\Delta < 0$, [6] has two complex conjugate roots $\omega/k = a + ib$ (say). This implies that a small perturbation will grow at a rate proportional to exp (kbt), indicating the onset of instability.†

In the limit of incompressible flow (c_L, $c_G \to \infty$), [6] becomes a quadratic in ω/k and the condition for stability is then simply that the relative velocity U_{GL} ($= U_G - U_L$) satisfies:

$$U_{GL} < \sqrt{(g(\rho_L - \rho_G)(h_G/\rho_G + h_L/\rho_L))}. \qquad [8]$$

This is the long wavelength stability condition derived by Wallis & Dobson (1973) using potential flow theory.

The critical relative velocities for instability predicted by [7] and [8] are plotted in figure 2 for air/water flows at 1 and 41 atmospheres ($\rho_* = 820$ and 20 respectively) in a 10 cm duct. For compressible flow the unstable region lies between the upper and lower (solid) curves. For the incompressible case the entire region above the lower (broken) curve is unstable. It is interesting to note that whereas the lower boundary of the unstable region is virtually unaltered by compressibility (and hence well represented by [8]) the upper boundary does not even exist for incompressible flows. The restabilisation at high relative velocities for compressible flows is explained as follows. At subsonic gas flows the instability is a long wavelength Kelvin–Helmholtz instability which is driven by the pressure reduction at a wave crest caused by the

†Because the growth rate is unbounded with respect to k, [5] are actually ill-posed for $\Delta < 0$ (see e.g. Richtmyer & Morton 1968). It can be shown that introduction of surface tension effects can remove the high frequency instabilities which lead to ill-posedness. However even with surface tension included [3] still predict instability to long wavelength pertubations when $\Delta < 0$.

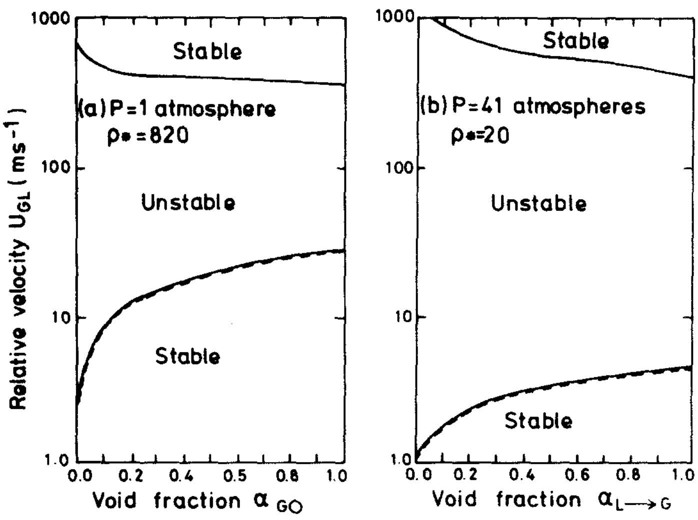

Figure 2. Relative velocities for onset of instability predicted by one-dimensional model of stratified flow. (Key: ———, Compressible flow[7]; – – – – – –, Incompressible flow[8]. Calculations given for an air–water system.)

constriction of the gas flow. At supersonic velocities the area reduction causes a *recovery* in gas pressure, which opposes wave growth. This ultimately stabilises the flow.

Equation [8] is compared with data on stability of stratified flow in section 4.

3. TWO-DIMENSIONAL ANALYSIS

A one-dimensional model of two-phase flow should give a correct description of wave motion in the limit when the wavelength is large compared with the length-scale of the flow structure across the duct. Thus a useful check on a one-dimensional approximation is provided by comparing predicted wave motion with the long wavelength limit of an exact solution. An exact analysis for small amplitude wave propagation is an infinite rectangular duct containing an inviscid stratified compressible two-phase flow is given briefly below.

3.1 *Governing equations and propagation analysis*

We again consider the geometry shown in figure 1. In fixed co-ordinates, the equations for conservation of mass and momentum for phase j are:

$$\left.\begin{aligned} &\frac{\partial \rho_j}{\partial t}+\frac{\partial}{\partial x}(\rho_j U_j)+\frac{\partial}{\partial y}(\rho_j V_j)=0 \\ &\frac{\partial U_j}{\partial t}+U_j\frac{\partial U_j}{\partial x}+V_j\frac{\partial U_j}{\partial y}+\frac{1}{\rho_j}\frac{\partial P_j}{\partial x}=0 \\ &\frac{\partial V_j}{\partial t}+U_j\frac{\partial V_j}{\partial x}+V_j\frac{\partial V_j}{\partial y}+\frac{1}{\rho_j}\frac{\partial P_j}{\partial y}=-g \end{aligned}\right\} \qquad [9]$$

where U and V are the particle velocities in the x and y directions respectively.

If the displacement of the interface at time t is denoted by $y = \eta(x, t)$, and η is sufficiently small, pressure and velocity boundary conditions at the interface can be written (see e.g. Lamb 1932),

$$\left.\begin{aligned} &P_L - P_G = -\sigma\frac{\partial^2\eta}{\partial x^2} \\ &V_j = \frac{\partial \eta}{\partial t}+U_j\frac{\partial \eta}{\partial x} \end{aligned}\right\} \text{ at } y=\eta(x,t)\,, \qquad [10]$$

where σ is the surface tension. The boundary conditions at the walls are:

$$\left.\begin{aligned} V_G = 0 \text{ at } y = h_G \\ V_L = 0 \text{ at } y = -h_L \end{aligned}\right\}. \qquad [11]$$

To calculate the motion of pressure and surface waves equations [9]–[11] are rewritten in co-ordinates moving vertically with the interface, and then solved for the motion of small perturbations of the form

$$\psi = \psi_0 + \psi'(y)\,e^{i(\omega t - kx)}\,.$$

Details of the analysis are given in the Appendix where it is shown that the required wave dispersion relationship is

$$\frac{\rho_* \omega_L^2}{\mu_L}\coth(\mu_L h_L)+\frac{\omega_G^2}{\mu_G}\coth(\mu_G h_G)=\left\{\frac{\sigma}{\rho_G}+g\left(\frac{\rho_*}{\mu_L^2}-\frac{1}{\mu_G^2}\right)\right\}k^2\,. \qquad [12]$$

For $\sigma = g = U_L = U_G = 0$ [12] is the dispersion relationship for pressure waves in a gas–liquid layer derived by Morioka & Matsui (1975). In the limit of incompressible flow ($c_j \to \infty$, $\mu_j \to k$) it reduces the usual result for surface wave propagation in a liquid of finite depth (Milne–Thomson 1968).

Equation [12] can be solved numerically for the phase velocity (ω/k). Figure 3 shows the results of illustrative calculations for a stationary air–water system at atmospheric pressure, ($\alpha_G = 0.5$). There is a single surface wave [curve (a)], and an infinite number of acoustic wave modes [(b), (c), (d), etc.] only the first four of which are shown. Acoustic wave modes of above lowest order [(c), (d), (e), etc.] are two-dimensional in character, with a cyclic variation of pressure and velocity across the duct width.

3.2 *Long wavelength limit*

For wavelengths large compared with the duct height [17] approaches the limit:

$$h_G\rho_*\mu_G^2\omega_L^2 + h_L\mu_L^2\omega_G^2 - \left\{\frac{\sigma}{\rho_G} + g\left(\frac{\rho_*}{\mu_L^2} - \frac{1}{\mu_G^2}\right)\right\} k^2 h_L h_G \mu_L^2 \mu_G^2 = 0\,. \qquad [13]$$

Equation [13] is a reasonable approximation for $kH \lesssim 0.4$ when $(\mu_j h_j)\coth(\mu_j h_j)$ differs from unity by less than 5 per cent. However, for these long waves travelling in ordinary liquids in pipes of practical sizes the term in σ is usually less than 1 or 2 per cent of the term in g. Thus, for conditions where [13] is valid it is sufficient to write

$$h_G\rho_*\mu_G^2\omega_L^2 + h_L\mu_L^2\omega_G^2 - gk^2h_Lh_G(\rho^*\mu_G^2 - \mu_L^2) = 0\,,$$

which is identical to the one-dimensional result [6] obtained from [3].

4. DISCUSSION

It has been seen that the one-dimensional two-fluid equations for a stratified flow with gravity, with surface tension ignored, predict wave propagation velocities which agree with an exact two-dimensional analysis in the appropriate low frequency limit. Also the one-dimensional equations are well-posed and stable over a range of (subsonic) relative velocities $U_{GL} < U_c$, which is approximated well by [8]. The predicted instability for larger relative motions is a long wavelength Kelvin–Helmholtz instability (in which liquid is driven towards a

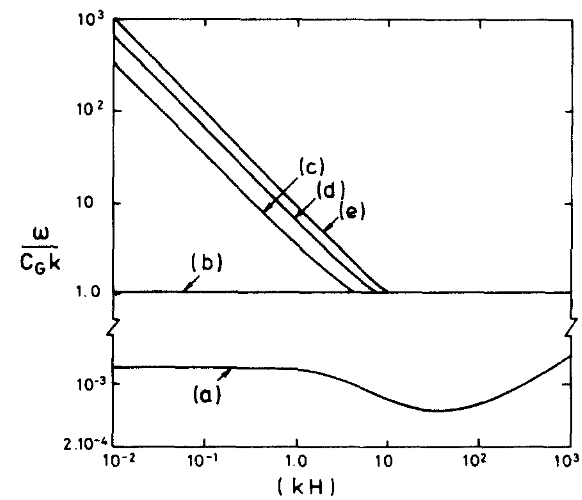

Figure 3. Velocity of acoustic and surface waves in a stratified air–water system at atmospheric pressure (calculations for $\alpha_G = 0.5$, $H = 10\,\text{cm}$, $\sigma = 73 \cdot 10^{-3}\,\text{Nm}^{-1}$, $c_G = 331\,\text{ms}^{-1}$, $c_L = 1460\,\text{ms}^{-1}$).

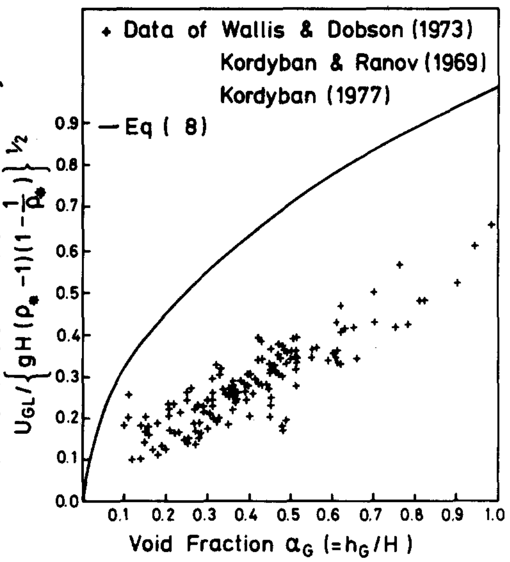

Figure 4. Comparison between [8] and observed relative velocities for the breakdown of horizontal stratified flow.

developing wave crest because of the local pressure reduction which results from the constriction of the gas flow). In the absence of gravity the instability is predicted for all finite relative velocities, with or without the inclusion of surface tension.

It is interesting to compare the implied stability limits with experimental observations of stratified flows. It has been observed that these flows are indeed only partially stable, and at sufficiently high interphase relative velocities transition to slug flow (in which the gas travels in long bubbles separated by slugs of liquid bridging the duct). Kordyban & Ranov (1967) suggested that this transition is a manifestation of the Kelvin–Helmholtz instability, and Wallis & Dobson (1973) observed that experimental measurements of the critical relative velocity could be satisfactorily correlated using the long wavelength instability criterion (8), if the r.h.s. is multiplied by a factor of 0.5. They suggested that this correction was probably needed to allow for modification of the ideal flow pressure field caused by separation at the wave crest. Figure 4 compares (8) with available measurements of U_{GL} for the onset of slugging in horizontal stratified flows. All data are for atmospheric pressure air–water systems and rectangular channels ($2 < H < 30$ cm).

It is seen that [8] over-predicts the transition velocity by a factor of about two, as expected. The implication is that when the governing two-fluid equations [1] are applied to a stratified flow they are stable and well-posed over a greater range of conditions than these flows are actually stable in practice.

It would be interesting to check that the flow restabilisation for super-sonic gas velocities also occurs in the manner suggested by the simple two-fluid calculations (section 2.2). It appears, however, that at present there are no high-speed flow data available to make any comparisons possible.

The observation that one-dimensional model is able to give an exact representation of the motion of long waves in an inviscid gas–liquid flow, and can predict flow stability over a realistic range, gives confidence that the adopted form of the conservation equations [1] is reasonable.

5. CONCLUSIONS

To help resolve uncertainties in what form is appropriate for the conservation equations of a one-dimensional two-fluid model a standard set of two-fluid equations has been used to calculate wave velocities and stability limits for the example of a frictionless stratified gas–liquid flow. It has been found for this case that in addition to predicting stability over a realistic range of conditions the two-fluid equations give wave velocities in agreement with the appropriate limit of an exact solution.

This work gives confidence in the use of these standard two-fluid equations for general application in transient two-phase flow analysis.

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APPENDIX

Calculation of wave velocities in a horizontal stratified flow

To calculate the velocity of small amplitude waves it is convenient to transform [1] into moving co-ordinates fixed in the gas liquid interface, defined by:

$$x' = x; \qquad y' = y - \eta(x, t); \qquad t' = t. \qquad [A1]$$

The transformed conservation equations for phase j are

$$\frac{\partial \rho_j}{\partial t'} - \eta_2 \frac{\partial \rho_j}{\partial y'} + \frac{\partial}{\partial x'}(\rho_j U_j) - \eta_1 \frac{\partial}{\partial y'}(\rho_j U_j) + \frac{\partial}{\partial y'}(\rho_j V_j) = 0,$$

$$\frac{\partial U_j}{\partial t'} - \eta_2 \frac{\partial U_j}{\partial y'} + U_j \frac{\partial U_j}{\partial x'} - U_j \eta_1 \frac{\partial U_j}{\partial y'} + V_j \frac{\partial U_j}{\partial y'} + \frac{1}{\rho_j}\frac{\partial P_j}{\partial x'} - \frac{1}{\rho_j}\frac{\partial P_j}{\partial y'} = 0,$$

$$\frac{\partial V_j}{\partial t'} - \eta_2 \frac{\partial V_j}{\partial y'} + U_j \frac{\partial V_j}{\partial x'} - U_j \eta_1 \frac{\partial V_j}{\partial y'} + V_j \frac{\partial V_j}{\partial y'} + \frac{1}{\rho_j}\frac{\partial P}{\partial y'} + g = 0, \qquad \text{[A2]}$$

where $\eta_1 = \partial\eta/\partial x$, $\eta_2 = \partial\eta/\partial t$.

In the unperturbed flow the interface is assumed to be horizontal and small density variations due to the hydrostatic pressure gradient are ignored. Thus

$$V_{j0} = \eta_0 = 0; \qquad P_{jo} = P^* - y\rho_{j0} g. \qquad \text{[A3]}$$

A perturbation is then imposed so that

$$\left.\begin{aligned} &U_j = U_{j0} + \epsilon u_j'; \quad V_j = \epsilon v_j' \\ &P_j = P_{j0} + \epsilon p_j'; \quad \rho_j = \rho_{j0} + \epsilon \rho_j' \\ &\eta = \epsilon \eta' \end{aligned}\right\} \qquad \text{[A4]}$$

where ϵ is small. Substituting [A4] and [A3] into [A2], using the fact that $\rho_j' = p_j'/c_j^2$, ignoring terms of above first order in ϵ, we get the following linearised equations

$$\left.\begin{aligned} &\frac{1}{c_j^2}\left[\frac{\partial p_j'}{\partial t'} + U_{j0}\frac{\partial p_j'}{\partial x'}\right] + \frac{\partial u_j'}{\partial x'} + \frac{\partial v_j'}{\partial y'} = 0 \\ &\frac{\partial u_j'}{\partial t'} + U_{j0}\frac{\partial u_j'}{\partial x'} + \frac{1}{\rho_j}\frac{\partial p_j'}{\partial x'} + g\frac{\partial \eta'}{\partial x'} = 0 \\ &\frac{\partial v_j'}{\partial t'} + U_{j0}\frac{\partial v_j'}{\partial x'} + \frac{1}{\rho_j}\frac{\partial p_j'}{\partial y'} + \frac{g}{c_j^2} p_j' = 0. \end{aligned}\right\} \qquad \text{[A5]}$$

The solutions of [A5] must satisfy boundary conditions obtained by inserting [A3] and [A4] in [10] and [11]

$$\left.\begin{aligned} &\left.\begin{aligned} p_L' - p_G' &= -\sigma\partial^2\eta'/\partial x^2 \\ v_j' &= \partial\eta'/\partial t' + U_{j0}\partial\eta'/\partial x' \end{aligned}\right\} \text{ at } y' = 0 \\ &v_G' = 0 \quad \text{at} \quad y' = h_G - \epsilon\eta' \\ &v_L' = 0 \quad \text{at} \quad y' = -h_L - \epsilon\eta'. \end{aligned}\right\} \qquad \text{[A6]}$$

We now consider plane wave perturbations

$$p_j'/\hat{p}_j(y) = u_j'/\hat{u}_j(y) = v_j'/\hat{v}_j(y) = \eta'/\hat{\eta} = \exp i(\omega t - kx),$$

where we have dropped the primed superscripts from the spatial co-ordinates. Substituting these forms into [A5] and [A7] gives a set of ordinary differential equations in the wave amplitudes

$$\left.\begin{aligned} i\omega_j\hat{p}_j - i\rho_j c_j^2 k\hat{u}_j &= -\rho_j c_j^2\,\mathrm{d}\hat{v}_j/\mathrm{d}y \\ -k\hat{p}_j + \rho_j\omega_j\hat{u}_j - \rho_j kg\hat{\eta} &= 0 \\ g\hat{p}_j/c_j^2 + i\omega_j\rho_j\hat{v}_j &= -\mathrm{d}\hat{p}_j/\mathrm{d}y, \end{aligned}\right\} \qquad [A7]$$

where

$$\left.\begin{aligned} \hat{v}_G = 0 \quad &\text{at} \quad y = h_G \\ \hat{v}_L = 0 \quad &\text{at} \quad y = -h_L \\ \left.\begin{aligned}\hat{p}_L - \hat{p}_G &= \sigma k^2\hat{\eta} \\ \hat{v}_k &= i\hat{\eta}\omega_k\end{aligned}\right\} \quad &\text{at} \quad y = 0. \end{aligned}\right\} \qquad [A8]$$

The first term on the left hand side of the third of [A7] arises because of the effects on the hydrostatic pressure gradient of small fluctuations in density and can be safely neglected. Solving the resultant equations it is easily shown that

$$\hat{p}_j = B_{1j}\,\mathrm{e}^{\mu_j y} + B_{2j}\,\mathrm{e}^{-\mu_j y} - \rho_j\hat{\eta}gk^2/\mu_j^2.$$

A corresponding relation exists for the $\hat{v}_j$'s. If these expressions are substituted in [A8] four homogeneous linear equations are obtained for the four integration constants B_{1j}, B_{2j}. The secular equation for this set is the required dispersion relation, which has the form

$$\rho_*\omega_L^2\mu_G \coth(\mu_L h_L) + \omega_G^2\mu_L \coth(\mu_G h_G) = \left[\frac{\sigma}{\rho_G} + g\left(\frac{\rho_*}{\mu_L^2} - \frac{1}{\mu_G^2}\right)\right] k^2\mu_L\mu_G. \qquad [A9]$$